Non-Linear Annular Networks

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Abstract

We show that the annular network $G(6,3)$ is recoverable, and we indicate how the proof might be generalized to $G(2n,n)$.

1 Introduction

We recall the basic definitions from [2].

1.1 Definition. A *graph with boundary* is a triple $\Gamma = (V, \partial V, E)$, where V is a set of vertices, $\partial V \subseteq V$ is a non-empty set of boundary vertices, and $E \subseteq V \times V$ is a symmetric, irreflexive relation on V. That is, $(i;i) \notin E$ for any $i \in V$, and $(i;j) \in E \iff (j;i) \in E$ for all $i, j \in V$. The *interior nodes* are those contained in $\operatorname{int} V = V \setminus \partial V$.

1.2 Definition. A *non-linear conductance network* is a pair (Γ, γ), where Γ is a graph with boundary, and γ is a map which assigns to each $(i;j) \in E$ a conductance function $\gamma_{i;j} : \mathbb{R} \to \mathbb{R}$, subject to the following constraints:

- $\gamma_{i;j}(-x) = -\gamma_{j;i}(x)$,
- $\gamma_{i;j}(0) = 0$,
- $\gamma_{i;j}$ is strictly increasing, and
- $\gamma_{i;j}$ is bijective.

Notice that the last two constraints together imply that $\gamma_{i;j}$ is continuous, and that this definition differs slightly from the one given in [2, Definition 2.2].

1.3 Definition. A *potential function* on a network (Γ, γ) is a function $u : V \to \mathbb{R}$. The *current* from vertex i to vertex j is $I_{i;j}(u) = \gamma_{i;j}(u(i) - u(j))$. The *total current* coming out of vertex i is given by

$$I_i(u) = \sum_{(i;j) \in E} I_{i;j}(u).$$

A potential function u is *harmonic* if $I_i(u) = 0$ for all $i \in \operatorname{int} V$.

2 Main Result

2.1 Definition. An *annular network* $G(m, n)$ is a network embedded in an annulus with m rays and n circles. We label the rays from 1 to m counter-clockwise starting from the north, and the circles from 1 to n starting from the innermost one. The interior vertices are those at the intersections of rays and circles, and we denote the vertex at the intersection of ray r and circle c by $v_{r,c}$. The exterior vertices are the endpoints of the rays, and we denote the inner and outer endpoints of ray r by $v_{r,0}$ and $v_{r,n+1}$ respectively. Also, we let $u_{i,j} = u(v_{i,j})$, where u is the potential function of $G(m, n)$. See $G(6, 3)$ in Figure 1 for example.

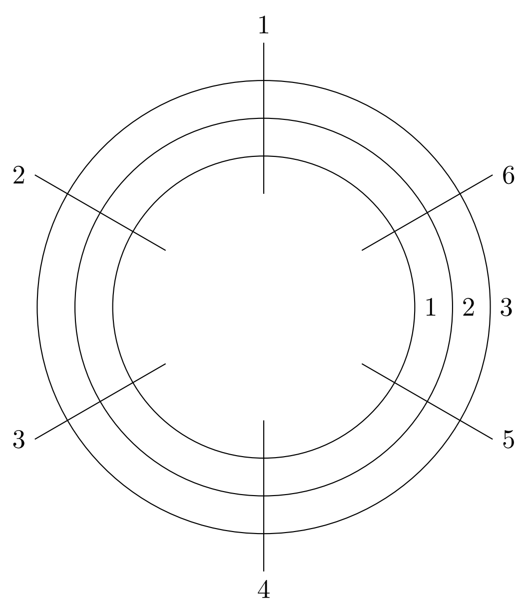

Figure 1: $G(6, 3)$ with its 6 rays and 3 circles labeled.

2.2 Lemma. [2, Corollary 6.3] The Dirichlet-to-Neumann map is well-defined for arbitrary non-linear conductance networks.

2.3 Theorem. The annular network $G(6, 3)$ is recoverable.

Proof. We begin by assigning the values shown in Figure 2, where the numbers in parentheses are currents. Right away, we see that $u_{2,1}, u_{3,1}, u_{4,1}, u_{5,1}, u_{6,1}$ are all zero by the harmonicity of u and the bijectivity of the conductivity function. Then for the same reasons, $u_{3,2}, u_{4,2}, u_{5,2}$ are also all zero, and then $u_{3,3} = 0$. Next, we look at $v_{1,1}, v_{2,2}, v_{3,3}, v_{5,3}, v_{6,2}$ following the argument in [1,

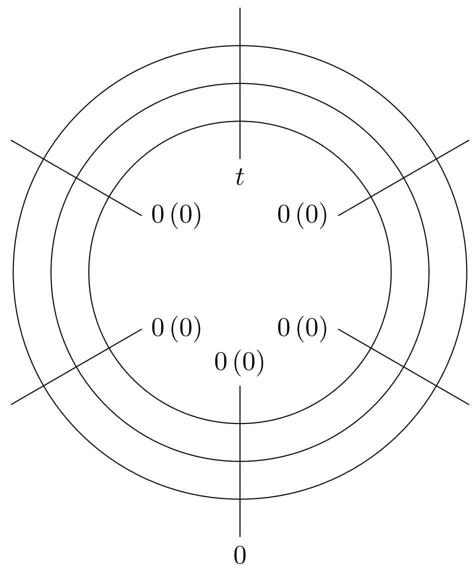

Figure 2: $G(6,3)$ with its initial conditions labeled.

Lemma 2.1]. Because u is harmonic, $u_{1,1}u_{2,2}$, $u_{2,2}u_{3,3}$, $u_{3,3}u_{5,3}$, $u_{5,3}u_{6,2}$, and $u_{6,2}u_{1,1}$ are all ≤ 0. Therefore, the product of all of these is also ≤ 0, but this is also a square, and hence ≥ 0. Thus, at least one of these voltages must be zero. But if one of them is zero, all of the others must also be zero because of the harmonicity of u.

Now we're going to show the current flows and the signs of the potentials. Suppose $t > 0$ for the sake of determining the direction of the current flows. By Lemma 2.1, for a given t, $I_{1,0}(t) = \gamma_{1,0;1,1}(t-0)$. Then, $\gamma_{1,0;1,1}$ is positive. So, by the harmonicity of the potential function, $\gamma_{1,1;1,2}$ is positive. Thus, since the conductivity function is bijective, $u_{1,2}$ is negative. So, since $u_{2,2} - u_{1,2}$ is positive, $\gamma_{2,2;1,2}(u_{2,2} - u_{1,2})$ is positive. By the same reasoning, $\gamma_{6,2;1,2}$ is positive. So, by harmonicity, $\gamma_{1,2;1,3}$ is positive. Therefore, $u_{1,3} < u_{1,2}$. So, $u_{1,3}$ is negative. By harmonicity, $\gamma_{2,3;2,2}$ and $\gamma_{6,3;6,2}$ are positive. As a result, $u_{2,3}$ and $u_{6,3}$ are positive. Hence, $\gamma_{2,3;1,3}(u_{2,3} - u_{1,3})$ and $\gamma_{6,3;1,3}(u_{6,3} - u_{1,3})$ are positive. By the harmonicity of the potential function $\gamma_{1,3;1,4}$ is positive. Accordingly, $u_{1,4} < u_{1,3}$. So, $u_{1,4}$ is negative. $\gamma_{2,3;3,3}(u_{2,3} - u_{3,3})$ is positive, since $u_{2,3} - u_{3,3}$ is positive. Then, by the same reasoning, $\gamma_{6,3;5,3}$ is positive. Therefore, by harmonicitiy, $\gamma_{2,4;2,3}, \gamma_{6,4;6,3}, \gamma_{5,3;5,4}, \gamma_{3,3;3,4}$ are positive. Hence, $u_{2,4} > u_{2,3}$, and $u_{6,4} > u_{6,3}$, so $u_{2,4}$ and $u_{6,4}$ are positive. $u_{5,4}$ and $u_{3,4}$ are negative, because $\gamma_{5,3;5,4}$ and $\gamma_{3,3;3,4}$ are positive. Therefore, all of the currents are completely determined by t.

Since $\gamma_{1,0;1,1}(t-0) = \gamma_{1,0;1,1}(t) = I_{1,0}(t)$, if t is varied over the real numbers the conductivity function, $\gamma_{1,0;1,1}(x)$ can be calculated by reading the current from the response map. By the symmetry of the graph, we can reassign the boundary values so that the conductance function can be found on each edge $(i;j) \in E$ where $i \in \partial V$.

Now consider $u_{2,3}$. Since the total current at $v_{2,4}$ and the potential $u_{2,4}$ can be read from the response map, $u_{2,3}$ can be calculated. By harmonicity, $\gamma_{2,3;3,3} = \gamma_{3,3;3,4} = -I_{3,4}(u)$, which can be measured. Then, since $u_{2,3}$ and $u_{3,3}$ are known, the conductance function on the edge, $(2,3;3,3)$, is known. $\gamma_{2,3;2,4}$ and $\gamma_{2,3;3,3}$ are known, and by symmetry, $\gamma_{2,3;1,3}$ is known. So by harmonicity, $\gamma_{2,3;2,2}$ is known. Since $u_{2,3}$ and $u_{2,2}$ are known, the conductance function $\gamma_{2,3;2,2}(x)$ is known.

By symmetry, $u_{1,2}$ is known, and by harmonicity, $\gamma_{2,2;1,2} = -\gamma_{2,2;2,3}$, which is known, so $\gamma_{2,2;1,2}(x)$ is known.

By symmetry, the conductance function is known for each edge $(i;j) \in E$.

□

3 Generalization to $G(2n, n)$

Consider the graph $G(2n, n)$ with $n \geq 4$.

3.1 Lemma. Let $u_{1,0} = t$, let $u_{2,0}, u_{3,0}, \ldots, u_{2n,0} = 0$, let $I_{2,0}, I_{3,0}, \ldots, I_{2n,0} = 0$, and let $u_{n+1,n+1} = 0$. Then,

$$\begin{array}{c} u_{1,1}, u_{2,1}, \ldots\ldots\ldots, u_{2n,1} \\ u_{2,2}, u_{3,2}, \ldots\ldots, u_{2n,2} \\ u_{3,3}, u_{4,3}, \ldots, u_{2n-3,3} \\ \vdots \\ u_{n+1,n} \end{array}$$

are all equal to zero.

Proof. By the harmonicity of the potential function, the zero potentials propogate, as in Theorem 2.3, such that $u_{2,1}, u_{3,1}, \ldots, u_{2n,1}$ all equal zero. On circle 1, there are $2n-1$ potentials of 0. All $2n-1$ of these potentials are on adjacent vertices with the middle vertex lying on the intersection between the circle 1 and ray $n+1$. Since u is harmonic, each time the index of the circle increases, two fewer vertices are determined to have potential zero. Since we have one fewer potential 0 on each side of the group of adjacent 0 potentials, the middle vertex will lie on the ray $n+1$. Since the first circle has $2n-1$ vertices with potential 0, the nth circle has $(2n-1)-(2(n-1)) = 2n-1-2n+2 = 1$ vertex with zero potential, since the group of vertices on the nth circle is centered around the $n+1$st ray, $u_{n+1,n} = 0$.

So, following the argument in [1, Lemma 2.1], $u_{1,1}u_{2,2}, u_{2,2}u_{3,3}, \ldots, u_{n-1,n-1}u_{n,n}$ and $u_{n,n}u_{n+2,n}, u_{n+2,n}u_{n+3,n-1}, \ldots, u_{2n,2}u_{1,1}$ are all ≤ 0. So, their product is ≤ 0, but is also ≥ 0 since it is a square. Thus, $u_{1,1}, u_{2,2}, \ldots, u_{n,n}$ and $u_{n+2,n}, u_{n+3,n-1}, \ldots, u_{2n,2}$ all equal zero, since if one is zero, then, by harmonicity, the rest are zero as well.

□

3.2 Conjecture. For the graph $G(2n, n)$, the voltages and currents are all determined by t.

3.3 Proposition. Suppose Conjecture 3.2 holds. Then, $G(2n, n)$ is recoverable.

Proof. Since $\gamma_{1,0;1,1}(t - 0) = \gamma_{1,0;1,1}(t) = I_{1,0}(t)$, if t is varied over the real numbers the conductivity function, $\gamma_{1,0;1,1}(x)$ can be calculated by reading the current from the response map. By the symmetry of the graph, we can reassign the boundary values so that the conductance function can be found on each edge $(i; j) \in E$ where $i \in \partial V$.

Consider $v_{n-1,n}$. Since $u_{n-1,n+1}$ is known by the response map, the total current on $v_{n-1,n+1}$ is known by the response map, and $\gamma_{n-1,n+1;n-1,n}$ is known, then $u_{n-1,n}$ is known. Then, since $\gamma_{n,n+1;n,n}$ is known, and the current on the edge, $(n, n; n-1, n)$ is equal to the current on $(n, n+1; n, n)$, $\gamma_{n,n;n-1,n}$ is known, and by symmetry, all edges on the nth and first circles are known. So, $\gamma_{n-1,n;n-2,n}$ is known. Then by harmonicity, the current on the edge $(n-1, n; n-1, n-1)$ is known. So, since $u_{n-1,n}$ and $u_{n-1,n-1}$ are also known, $\gamma_{n-1,n;n-1,n-1}$ is known. By symmetry, all the $\gamma_{i,n;i,n-1}$ and $\gamma_{i,1;i,2}$, where $i = 1, 2, \ldots, 2n-1, 2n$ are known.

Each vertex $v_{i,i} \in \text{int}V$ has four edges adjacent to it, two of the edges adjacent to each $v_{i,i}$ have current zero, so, by harmonicity, the other two must have currents of equal magnitude, but opposite sign. So by following the procedure above, the conductances on the edges adjacent to the $v_{i,i}$s can be found, so by symmetry, all of the conductances and all of the potentials are known for graphs with $2n$ rays and n circles. □

References

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